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The Buck–Sukumar model described in terms of $su(2) \otimes su(1,1)$ coherent states

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Abstract

The Buck–Sukumar model, which describes an assembly of A identical two-level atoms in interaction with a monochromatic radiation field, is investigated using $su(2) \otimes su(1,1)$ coherent states, in the framework of conventional mean-field many-body approaches. In particular, the super-radiant phase transition is studied. We find that results based on the mean-field method compare favorably with exact results. We also find that the results are much improved if the constant of motion of the model is implemented exactly, with the help of appropriate projection techniques, instead of being implemented only in the average. Since the Hamiltonian of the Buck–Sukumar model is unbounded from below, i.e., it lacks a ground state, a stabilized version of the model is also studied.

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1. Introduction

The one photon Jaynes–Cummings (JC) model [1] is an important model of quantum optics and quantum electronics. It describes a two-level atom in interaction with a monochromatic radiation field. The Hamiltonian of the model is given by

$$H_0 = \omega_f a^\dagger a + \omega_a s_z + \lambda(s_+ a + s_- a^\dagger), \tag{1}$$

where $s_z, s_\pm$ are operators satisfying the commutation relations of the $su(2)$ algebra, $[s_z, s_\pm] = \pm s_\pm$ and $[s_+, s_-] = 2s_z$, and $a^\dagger, a$ are respectively creation and destruction operators of the photon, satisfying the boson commutation relation $[a, a^\dagger] = 1$, ω_f is the energy of each photon, ω_a is the energy splitting between the relevant atomic states and λ is a real coupling constant. The operators $s_z, s_\pm$ are 2×2 matrices related in the standard way to the Pauli spin matrices. This is an exactly solvable model [2]. Despite its simple form, it shows important quantum features as, for instance, the collapse and revival of atomic inversion [3] and the squeezing of the radiation field [4]. This model illustrates important physical effects such as the behavior of the ^{85}Rb atom micromaser [5], the ^{138}Ba atom microlaser [6] and spin polarized neutrons subject to a magnetic field [7]. In [3], Eberly *et al* studied the average value of the atomic excitation energy, $\langle s_z(t)\rangle$, in the vicinity of a resonance ($\omega_f = \omega_a$). In [8], Buck and Sukumar modified the interaction term in Hamiltonian (1) in such a way that the Heisenberg equations of motion could be solved exactly and the expression of $\langle s_z(t)\rangle$ could be evaluated in closed form. They proposed the Hamiltonian

$$H = \omega a^\dagger a + \omega s_z + \lambda(s_+ a\sqrt{a^\dagger a} + s_-\sqrt{a^\dagger a}a^\dagger), \tag{2}$$

where $\omega = \omega_a = \omega_f$. In [9–11], analogous models of interacting matter-radiation, involving the description of the electromagnetic field in terms of similarly deformed bosons, have been investigated. These models are relevant for the description of the phenomenon of super-radiance, first considered by Dicke [12].

The Buck–Sukumar (BS) model may be generalized for the case of an assembly of A identical two-level atoms. It is enough to substitute into (2) the operators $s_z, s_\pm$ by the total spin operators

$$S_z = \sum_{j=1}^{A} s_z^{(j)}, \qquad S_\pm = \sum_{j=1}^{A} s_\pm^{(j)}, \tag{3}$$

where $s_z^{(j)}, s_\pm^{(j)}$ are operators associated with the j th atom. The so-called Dicke model [12] is obtained from the JC model by the same procedure. It has been shown that coherent states [13] are good choices for trial states, in variational approaches to the description of the behavior of a great variety of quantum systems. For example, they have been recently used to describe the ground-state and RPA energies of the two-level pairing model and the Lipkin model [14–16]. In the same spirit, we use now coherent states to investigate some properties of the BS model.

2. The model

For simplicity, we restrict our attention to the following form of the Hamiltonian of the BS model:

$$H = a^\dagger a + S_z + \lambda(S_+R_- + S_-R_+), \tag{4}$$

where $S_z, S_\pm$ are given by (3) and $R_- = a\sqrt{a^\dagger a}$, $R_+ = \sqrt{a^\dagger a}a^\dagger$. In (4), we have taken $\omega = 1$. In the following, all energies will be given in units of ω. The operators $S_z, S_\pm$ satisfy the commutation relations of the $su(2)$ algebra,

$$[S_z, S_\pm] = \pm S_\pm, \qquad [S_+, S_-] = 2S_z.$$

The Casimir operator $\mathbf{S}^2 = S_\mp S_\pm + S_z^2 \pm S_z$ is a constant of motion. Its eigenvalues $S(S+1)$ characterize the assembly of atoms described by H, in the sense that $2S$ represents the number of assembled atoms. Since the spectrum of H does not depend on the sign of the coupling constant, we assume that $\lambda \geqslant 0$. The operators $R_\pm$ and $R_0 = (a^\dagger a + 1/2)$ satisfy the commutation relations of the $su(1,1)$ algebra,

$$[R_0, R_\pm] = \pm R_\pm, \qquad [R_+, R_-] = -2R_0. \tag{5}$$

The corresponding Casimir operator reads

$$Q = -R_\mp R_\pm + R_0^2 \pm R_0. \tag{6}$$

In general, its eigenvalues are of the form $k(k-1)$, $k \in \mathbb{C}$. However, in the boson realization under consideration, $Q = -1/4$, so that $k = 1/2$.

The Hilbert space of the model is $\mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}_B$, where $\mathcal{H}_F$ and $\mathcal{H}_B$ are, respectively, the fermion (assembly of two-level atoms) and boson (photon system) Hilbert subspaces. An orthonormal basis of $\mathcal{H}$ is formed by the kets

$$|S, m\rangle \otimes |n\rangle \tag{7}$$

where $|S, m\rangle$ are (normalized) eigenstates of $\mathbf{S}^2$ and S_z, i.e., $\mathbf{S}^2|S, m\rangle = S(S+1)|S, m\rangle$, $S_z|S, m\rangle = m|S, m\rangle$ and $|n\rangle$ are (normalized) eigenstates of the boson number operator, $a^\dagger a|n\rangle = n|n\rangle$, with $S = 0, 1/2, 1, 3/2, \ldots, m = -S, -S+1, \ldots, S$ and $n = 0, 1, 2, \ldots$. The operator

$$C = a^\dagger a + S_z \tag{8}$$

is a constant of motion, since $[C, (S_+R_- + S_-R_+)] = 0$. The eigenvalues of C, denoted by c, are integers or half-integers. If $S \geqslant 0$ and c are integers or half-integers and $c + S$ and $n \geqslant 0$ are integers satisfying $c + S \geqslant n \geqslant c - S$, the kets,

$$|S, c-n\rangle \otimes |n\rangle, \tag{9}$$

are eigenvectors of C associated with the eigenvalue c. By $\mathcal{H}^{(S,c)}$ we denote the subspace spanned by these kets, for fixed S and c. Obviously, we must have $c \geqslant -S$. For $c \geqslant S$, $\mathcal{H}^{(S,c)}$ has dimension $2S+1$. If $c < S$ the dimension is $2S + 1 + (c - S) = S + c + 1$. By $E_0^{(S,c)}$ we denote the lowest eigenvalue of H for the specified S and c. Since the subspaces $\mathcal{H}^{(S,c)}$ are left invariant by the Hamiltonian and are finite dimensional, the determination of the exact spectrum of the Hamiltonian is trivial.

We are interested in the eigenspace of $\mathbf{S}^2$ associated with the eigenvalue $S(S+1)$, which is the direct sum of the subspaces $\mathcal{H}^{(S,c)}$,

$$\mathcal{H}^{(S)} = \bigoplus_c \mathcal{H}^{(S,c)}, \tag{10}$$

where it is implicit that c runs over the meaningful values.

3. Trial states

3.1. $su(2) \otimes su(1,1)$ *coherent states*

We will investigate the ground-state properties and the critical behavior of our system using suitable trial states. Since the Hamiltonian is expressed in terms of the generators of the $su(2)$ and $su(1,1)$ algebras, it is natural to consider the $su(2) \otimes su(1,1)$ coherent states as the desired trial states.

Let $|0\rangle_F \equiv |S, -S\rangle \in \mathcal{H}_F$, where $S_-|S, -S\rangle = 0$, $S_z|S, -S\rangle = -S|S, -S\rangle$ and $|0\rangle_B \equiv |0\rangle \in \mathcal{H}_B$ where $a|0\rangle = 0$. Our trial state will be, therefore, the coherent state

$$|\psi) = \mathrm{e}^{zS_+}\,\mathrm{e}^{\xi R_+}|0) \in \mathcal{H}^{(S)}, \qquad z, \xi \in \mathbb{C}, |\xi| < 1 \tag{11}$$

where $|0) = |0\rangle_F \otimes |0\rangle_B$ is the global vacuum.

The relevant average values read

$$\frac{(\psi|S_z|\psi)}{(\psi|\psi)} = S\frac{|z|^2 - 1}{|z|^2 + 1}, \qquad \frac{(\psi|S_+|\psi)}{(\psi|\psi)} = \frac{(\psi|S_-|\psi)^*}{(\psi|\psi)} = 2S\frac{z^*}{|z|^2+1}, \tag{12}$$

$$\frac{(\psi|a^\dagger a|\psi)}{(\psi|\psi)} = \frac{|\xi|^2}{1-|\xi|^2}, \qquad \frac{(\psi|R_+|\psi)}{(\psi|\psi)} = \frac{(\psi|R_-|\psi)^*}{(\psi|\psi)} = \frac{\xi^*}{1-|\xi|^2}. \tag{13}$$

Thus, from (12) and (13), the energy expectation value in state $|\psi)$ reads

$$\mathcal{E} = \frac{(\psi|H|\psi)}{(\psi|\psi)} = \frac{|\xi|^2}{1-|\xi|^2} + S\frac{|z|^2-1}{|z|^2+1} - 4S\lambda\frac{|z||\xi|}{(|z|^2+1)(1-|\xi|^2)}. \tag{14}$$

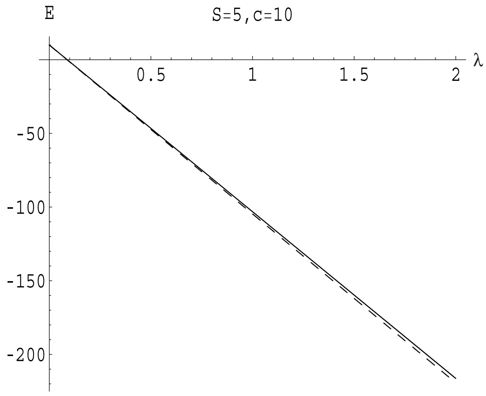


Figure 1. The lowest energy for $S = 5$ and $c = 10$. The thin line is the exact result; the dashed line is the minimum energy of the state $|\psi)$; the minimum energy of the state $|\psi_p)$ is represented by a line which practically coincides with the exact result.

This expectation value has been optimized with respect to the phases of z and ξ by choosing $(\arg z - \arg \xi) = \pi$.

Since the state $|\psi)$ does not belong to the subspace $\mathcal{H}^{(S,c)}$, we implement in the average the conservation of C,

$$\frac{(\psi|a^\dagger a + S_z|\psi)}{(\psi|\psi)} = \frac{|\xi|^2}{1-|\xi|^2} + S\frac{|z|^2-1}{|z|^2+1} = c. \tag{15}$$

In figure 1, we compare the exact ground-state energy with its variational estimate for the case $S = 5$ and $c = 10$. The agreement is rather good. In figure 1, the exact ground-state energy appears slightly above its variational estimate, apparently contradicting the Ritz theorem. This happens because some components of the state $|\psi)$ lie outside the subspace $\mathcal{H}^{(S,c)}$.

3.2. Projection on the physical subspace

In order to improve the description provided by the state $|\psi)$ (equation (11)) we project it on the subspace $\mathcal{H}^{(S,c)}$. The projected state reads

$$|\psi_p) = \sum_{n=n_0}^{c+S} \frac{\rho^n}{n!(c+S-n)!} S_+^{c+S-n} R_+^n |0), \tag{16}$$

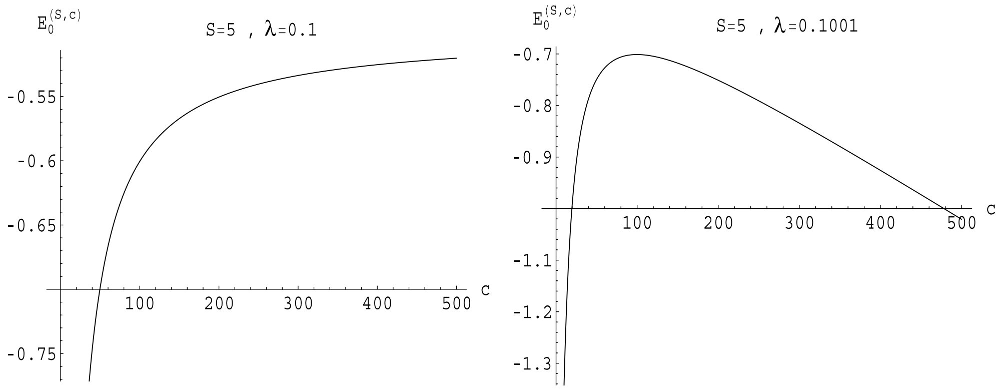
where $\rho = \xi/z$ and $n_0 = \mathrm{Max}((c-S), 0)$. For $c \geqslant S$, which happens when the number of photons exceeds the number of atoms, we have

$$\frac{(\psi_p|S_+R_- + S_-R_+|\psi_p)}{(\psi_p|\psi_p)} = 4S\frac{(c+S)|\rho|^3 + (c-S+1)|\rho|}{(1+|\rho|^2)^2}\cos(\alpha), \tag{17}$$

where $\alpha = \arg \rho$. Since $(\psi_p|a^\dagger a + S_z|\psi_p) = c(\psi_p|\psi_p)$ the energy expectation value of $|\psi_p)$ reads

$$\frac{(\psi_p|H|\psi_p)}{(\psi_p|\psi_p)} = c - 4S\lambda\frac{(c+S)|\rho|^3 + (c-S+1)|\rho|}{(1+|\rho|^2)^2}, \tag{18}$$

after optimization with respect to α. In figure 1, the minimum of the estimated energy is represented for $S = 5$ and $c = 10$. It almost coincides with the exact value, the difference between both curves being undetectable. Also for $c \leqslant S$, the performance of the projected state $|\psi_p)$ is superior to that of the coherent state $|\psi)$.



(*a*) (*b*)

Figure 2. $E_0^{(S,c)}$ as a function of c, for $S = 5$ and λ fixed. In (*a*), $\lambda = 0.1$ (normal phase). In (*b*), $\lambda = 0.1001$ (super-radiant phase).

4. BS model completed

We have restricted our attention to states characterized by a specific value c of the constant of motion C. When we relax this restriction and compare the properties of states characterized by different values of c, for the same S, we find, with some surprise, that the energy has a lower bound only if $\lambda \leqslant 1/(2S)$. In this case, $(\psi|H|\psi)/(\psi|\psi) \geqslant -S$. Then, a ground state exists such that $E_0^{(S,c)} = -S$, which occurs for $c = -S$. If $\lambda > 1/(2S)$, the Hamiltonian is unbounded from below, which may be physically unacceptable. Then, there is no value of c for which the energy is minimal. Increasing c indefinitely leads to an indefinite decrease of the energy. In this sense, we say that the model is *incomplete* for $\lambda > 1/(2S)$. As figure 2 shows, for a value of λ below $1/(2S)$, when c increases, the lowest energy value also increases. This no longer happens for a value of λ above $1/(2S)$. Then, the lowest energy decreases indefinitely when c increases after a certain critical value. This effect illustrates dramatically the phase transition from the normal to the so-called super-radiant phase.

We stabilize the model by adding to the Hamiltonian a term quadratic in C. We now consider the stabilized Hamiltonian

$$H_\epsilon = a^\dagger a + S_z + \epsilon(a^\dagger a + S_z)^2 + \lambda(S_+R_- + S_-R_+), \tag{19}$$

where ϵ is a positive real parameter, which is small enough so that the desirable physical features of the original Hamiltonian are not destroyed. In the following, we will refer to this Hamiltonian as the *completed Buck–Sukumar* (cBS) *model*. The original Hamiltonian and the stabilized one have the same eigenstates, their eigenvalues being displaced by ϵc^2. The conclusions we draw for the stabilized Hamiltonian have, therefore, a counterpart in the original one.

We investigate the behavior of the minimum energy of the stabilized Hamiltonian for a given c, when c is varied. In figure 3(*a*), the case $S = 5$, $\epsilon = 0.01$ and $\lambda = 0.175$ is presented. The coupling constant is somewhat above the critical value and the minimum energy occurs already for a rather high value of c. The performance of the projected state is remarkable. Below the critical value, the minimum energy occurs for $c = -S$. An important aspect of the BS model is the phase transition for $\lambda = 1/(2S)$. In figure 3(*b*) the phase transition for the cBS model is clearly exhibited, for $\epsilon = 0.01$. The stabilizing term pushes the critical point to a somewhat higher value of λ. When the value of ϵ decreases to 0, the critical value of λ approaches $1/(2S)$.

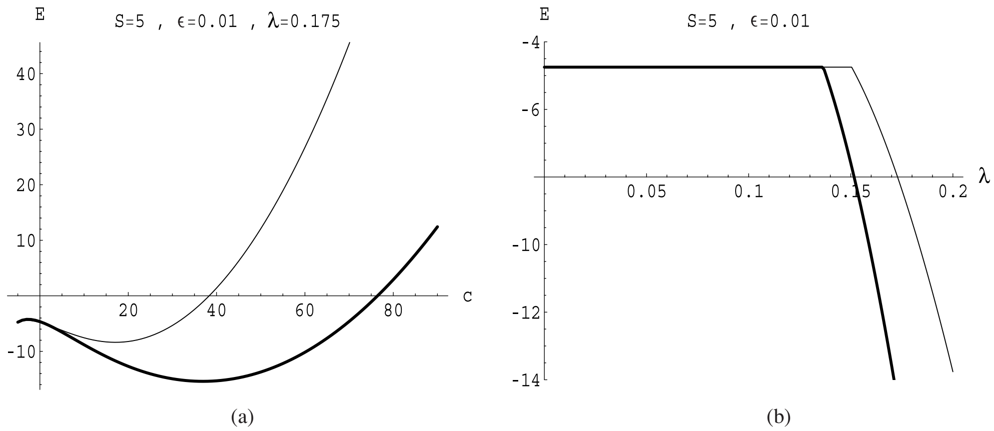


Figure 3. Properties of the cBS model. (*a*) $E_0^{(S,c)}$ as a function of c for $S = 5$ and $\lambda = 0.175$, (*b*) ground-state energy for $S = 5$. The thick line is the exact result, on which the estimate based on the projected state $|\psi_p)$ is superimposed. The thin line is the variational estimate based on the coherent state $|\psi)$.

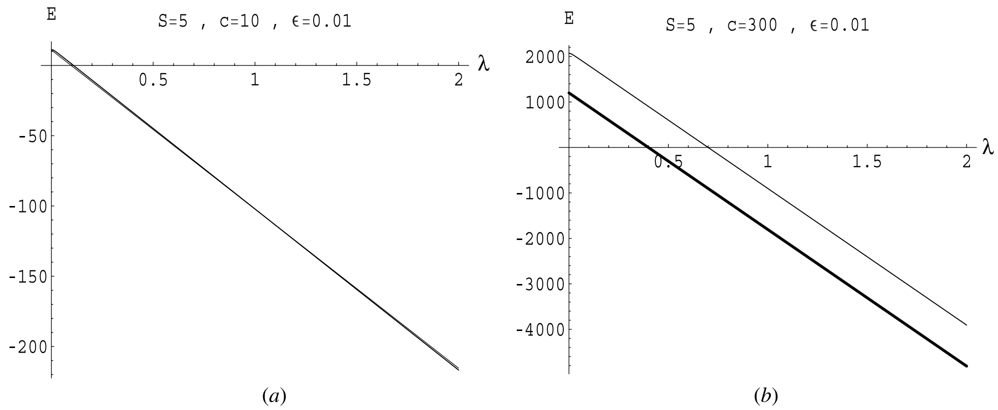
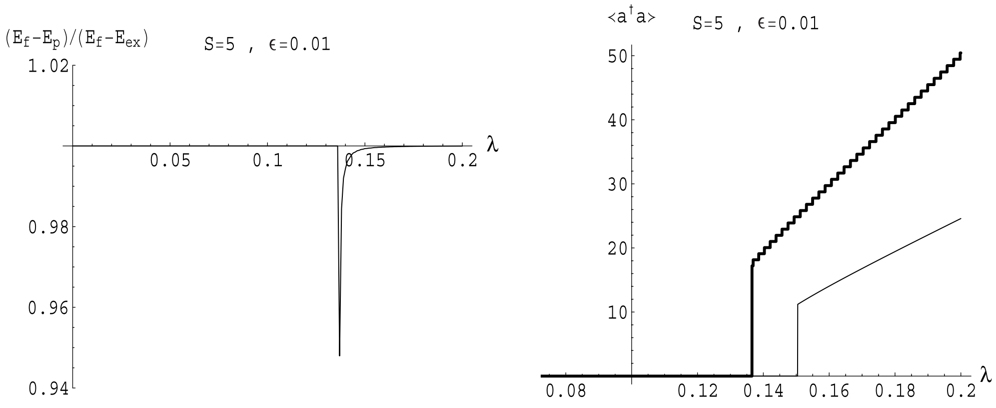


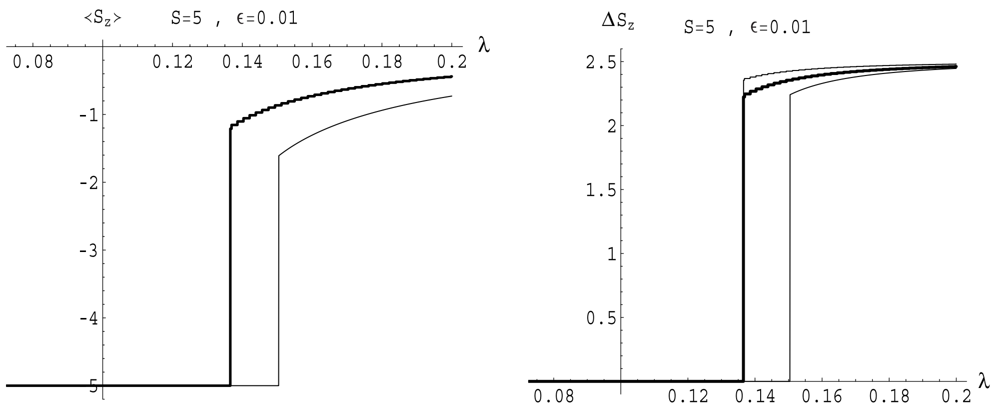
Figure 4. Properties of the cBS model. Lowest energy for $S = 5$ and fixed values of c. (*a*) For $c = 10$, the exact result and the minimum energy for the coherent state $|\psi)$ are represented by lines which practically coincide. (*b*) For $c = 300$, the upper line corresponds to the coherent state $|\psi)$ and the lower one to the exact result superimposed on the result for the projected state $|\psi_p)$; the discrepancy, due to fluctuations in the photon number, is noticeable. In both cases, the performance of the projected state $|\psi_p)$ is remarkable, the corresponding line being superimposed on the exact one.

In figure 3(*b*), we also compare the estimated minimum energy based on the coherent state $|\psi)$ with the exact ground-state energy, for $S = 5$ and $\epsilon = 0.01$. We see that the agreement is excellent before the transition and reasonable after it. This happens because, within a given subspace $\mathcal{H}^{(S,c)}$, the performance of the coherent state $|\psi)$ is very good, as shown in figure 4(*a*), if c is not too high. However, it becomes gradually worse when c increases, and a sizeable discrepancy is already noticeable in figure 4(*b*), due essentially to the fluctuations in c which are present in the coherent state $|\psi)$. Now, the ground state of the cBS model occurs for a rather large value of c, for which the performance of $|\psi)$ is reasonable but no longer excellent. Nevertheless, this deficiency of $|\psi)$ is corrected in the projected state $|\psi_p)$, so that the variational estimate of the ground-state energy based on this state is in very good agreement



(a) (b)

Figure 5. Properties of the cBS model: correlation energy and order parameter. (a) Ratio between the correlation energy estimated, in the projection method, by $E_f - E_p$ and the exact correlation energy $E_f - E_{\mathrm{ex}}$; here, E_f is the ground-state energy estimated by the coherent state $|\psi)$, E_p is the ground-state energy estimated by the projected state $|\psi_p)$ and E_{ex} is the exact ground-state energy. (b) Behavior of the number of photons $\langle a^\dagger a\rangle$, regarded as order parameter. The lower line, in the super-radiant phase, refers to the coherent state $|\psi)$, while the upper line represents the exact values, on which the projected state $|\psi_p)$ results are superimposed.



(a) (b)

Figure 6. (a) Average value of S_z. (b) $\Delta S_z = \langle S_z^2\rangle - \langle S_z\rangle^2$.

with the exact result also in the super-radiant phase. The same applies to the estimates of the lowest energy values for fixed S and c. In figure 5(a) we compare, for different values of the coupling constant λ, the correlation energy estimated by the projection method as $(E_f - E_p)$ with the exact correlation energy $(E_f - E_{\mathrm{ex}})$, where E_f is the ground-state energy estimated by the coherent state $|\psi)$, E_p is the ground-state energy estimated by the projected state $|\psi_p)$ and E_{ex} is the exact ground-state energy. The ratio of these quantities is 1.0, except in the immediate vicinity of the critical point, where it is 0.948. In figure 5(b), the behavior of the number of photons $\langle a^\dagger a\rangle$, regarded as order parameter, for different values of the coupling constant λ, shows a first-order transition. The performance of the projected state, which leads to results coinciding with the exact ones within three figures, is remarkable, the effect of photon number quantization being clearly exhibited in both cases. In figure 6(a) we compare, for different values of the coupling constant λ, the average value of S_z for the coherent state $|\psi\rangle$ (thin line), with the corresponding value estimated with the projected state $|\psi_p\rangle$, which appears superimposed on the exact value (thick line). In figure 6(*b*), we compare, for different values of the coupling constant λ, the variance $\Delta S_z = \langle S_z^2\rangle - \langle S_z\rangle^2$ for the coherent state $|\psi\rangle$ (lower thin line), with the corresponding value estimated with the projected state $|\psi_p\rangle$ (upper thin line), which almost coincides with the exact value (thick line). Note the remarkable jumps, reflecting a first-order phase transition, at the super-radiance critical point.

As it has been observed already, we also note that the critical point predicted by the projection method coincides with the exact critical point and lies slightly below the critical point predicted by the coherent state $|\psi\rangle$.

5. Conclusions

The Buck–Sukumar model was investigated using $su(2) \otimes su(1, 1)$ coherent states, in the framework of conventional mean-field many-body approaches. The super-radiant phase transition was studied. We found that variational results obtained in the framework of mean-field approaches compare favorably with exact results. We also found that the obtained results are much improved if the constant of motion of the model is implemented exactly, with the help of appropriate projection techniques, instead of being implemented only in the average.

In order to circumvent a drawback of the original BS model, which lacks a ground state, that is, the spectrum of its Hamiltonian has the undesirable feature of being unbounded from below, a stabilized version was proposed and studied, confirming that the investigated coherent states provide a good description of the model.

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